SCHWARZIAN NORMS AND TWO-POINT DISTORTION

Martin Chuaqui, Peter Duren, William Ma,
Diego Mejía, David Minda, and Brad Osgood

Abstract. An analytic function f with Schwarzian norm $\|\mathcal{S}f\| \le 2(1+\delta^2)$ is shown to satisfy a pair of two-point distortion conditions, one giving a lower bound and the other an upper bound for the deviation. Conversely, each of these conditions is found to imply that $\|\mathcal{S}f\| \le 2(1+\delta^2)$. Analogues of the lower bound are also developed for curves in $\mathbb{R}^n$ and for canonical lifts of harmonic mappings to minimal surfaces.

§1. Introduction.

A well known theorem of Nehari [16] states that if the Schwarzian derivative $\mathcal{S}f = (f''/f')' - \frac{1}{2}(f''/f')^2$ of an analytic locally univalent function f satisfies the inequality

$$|\mathcal{S}f(z)| \le \frac{2}{(1-|z|^2)^2} \tag{1}$$

for all points z in the unit disk $\mathbb{D}$, then f is univalent in $\mathbb{D}$. The result is best possible, since for any $\delta > 0$ the weaker condition

$$|\mathcal{S}f(z)| \le \frac{2(1+\delta^2)}{(1-|z|^2)^2}\,, \qquad z \in \mathbb{D}\,, \tag{2}$$

admits functions f with infinite valence. However, such functions are uniformly locally univalent in the sense that any two distinct points where f assumes equal values are uniformly separated in the hyperbolic metric

$$d(\alpha,\beta) = \frac{1}{2}\log\frac{1+\rho(\alpha,\beta)}{1-\rho(\alpha,\beta)}\,, \qquad \text{where} \quad \rho(\alpha,\beta) = \left|\frac{\alpha-\beta}{1-\overline{\alpha}\beta}\right|\,.$$

More precisely, if f satisfies the inequality (2) for some constant $\delta > 0$, then $d(\alpha,\beta) \ge \pi/\delta$ for any pair of points α and β in $\mathbb{D}$ where $f(\alpha) = f(\beta)$ but $\alpha \ne \beta$.

1991 *Mathematics Subject Classification.* Primary 30C55, Secondary 31A05, 53A10.

Key words and phrases. Univalent function, Schwarzian derivative, Schwarzian norm, hyperbolic metric, two-point distortion, harmonic mapping, minimal surface.

The first, second, and sixth authors are supported by Fondecyt Grant # 1071019. The third, fourth, and fifth authors were participants in the Taft Research Seminar, 21st Century Geometric Analysis at the University of Cincinnati; the third and fifth authors gratefully acknowledge partial support from the Taft Research Center.

Moreover, the separation constant π/δ is best possible. This result is essentially due to B. Schwarz [17]. A proof and further discussion can be found in [4]. Generalizations to Nehari functions other than $p(x) = (1 - x^2)^{-2}$ are given in [4] and [5].

The Schwarzian norm of an analytic locally univalent function f is defined by

$$\|\mathcal{S}f\| = \sup_{z\in\mathbb{D}} (1 - |z|^2)^2 |\mathcal{S}f(z)| \,.$$

Thus Nehari's theorem says that f is univalent if $\|\mathcal{S}f\| \le 2$, whereas the theorem of Schwarz says it is uniformly locally univalent if $\|\mathcal{S}f\| \le 2(1+\delta^2)$ for some constant $\delta > 0$.

Chuaqui and Pommerenke [9] gave a quantitative version of Nehari's theorem by showing that the condition $\|\mathcal{S}f\| \le 2$ implies that f has the two-point distortion property

$$\Delta_f(\alpha,\beta) = \frac{|f(\alpha) - f(\beta)|}{\{(1-|\alpha|^2)|f'(\alpha)|\}^{1/2}\,\{(1-|\beta|^2)|f'(\beta)|\}^{1/2}} \ge d(\alpha,\beta) \qquad (3)$$

for all points $\alpha, \beta \in \mathbb{D}$. Conversely, they found that if f satisfies (3), then $\|\mathcal{S}f\| \le 2$. Thus the distortion property (3) actually characterizes functions in the Nehari class.

In the present paper we show more generally that for any $\delta > 0$ the analytic functions with Schwarzian norm $\|\mathcal{S}f\| \le 2(1 + \delta^2)$ are characterized by the local distortion property

$$\Delta_f(\alpha,\beta) \ge \frac{1}{\delta}\,\sin(\delta\, d(\alpha,\beta))\,, \qquad \alpha,\beta\in\mathbb{D}\,,\quad d(\alpha,\beta) \le \frac{\pi}{\delta}\,. \qquad (4)$$

Note that the lower bound equals zero, as it must, when $d(\alpha,\beta) = 0$ or π/δ. Observe also that as $\delta \to 0$, the inequality (4) reduces to (3).

We also show that for any constant $\delta > 0$ an analytic function f has Schwarzian norm $\|\mathcal{S}f\| \le 2(1 + \delta^2)$ if and only if

$$\Delta_f(\alpha,\beta) \le \frac{1}{\sqrt{2+\delta^2}}\,\sinh\bigl(\sqrt{2+\delta^2}\,d(\alpha,\beta)\bigr)\,, \qquad \alpha,\beta\in\mathbb{D}\,. \qquad (5)$$

As a corollary, we can draw the rather surprising conclusion that for any constant $\delta > 0$ and any analytic function f, the upper bound (5) holds for all points $\alpha, \beta \in \mathbb{D}$ if and only if the lower bound (4) holds for all $\alpha, \beta \in \mathbb{D}$ with $d(\alpha,\beta) \le \pi/\delta$. Also, an analytic function f satisfies $\Delta_f(\alpha,\beta) \le \frac{1}{\sqrt{2}}\sinh\bigl(\sqrt{2}\,d(\alpha,\beta)\bigr)$ for all $\alpha, \beta \in \mathbb{D}$ if and only if f is univalent and $\|\mathcal{S}f\| \le 2$.

The final section of the paper develops a generalization of the lower bound (4) for canonical lifts of harmonic mappings to minimal surfaces.

§2. A basic lemma.

The proofs make essential use of a comparison lemma for solutions of differential equations, which we now state.

Comparison Lemma. *Let $Q(x)$ be continuous and $Q(x) > 0$ for $x \in [0,1)$. Let $v(x)$ and $w(x)$ be defined as the solutions of*

$$v''(x) + Q(x)v(x) = 0\,, \qquad v(0) = 0\,,\ v'(0) = 1$$

and

$$w''(x) - Q(x)w(x) = 0\,, \qquad w(0) = 0\,,\ w'(0) = 1\,,$$

respectively. Suppose that $v(x) > 0$ in an interval $(0,\xi)$, where $0 < \xi \le 1$. Let $p(z)$ be analytic and satisfy $|p(z)| \le Q(|z|)$ for all $z \in \mathbb{D}$. Then the solution of

$$u''(z) + p(z)u(z) = 0\,, \qquad u(0) = 0\,,\ u'(0) = 1$$

satisfies the inequalities

$$v(|z|) \le |u(z)| \quad \text{for } |z| < \xi\,, \qquad |u(z)| \le w(|z|)\,, \quad \text{for all } z \in \mathbb{D}\,.$$

It is clear that $w(x) > 0$ for all $x \in (0,1)$, since the differential equation implies that $w''(x) \ge 0$. On the other hand, $v''(x) \le 0$ and so it is possible that $v(x) = 0$ for some $x \in (0,1)$.

The upper inequality $|u(z)| \le w(|z|)$ was proved and applied by Essén and Keogh [12]. Herold [13] had previously obtained a more general result for differential equations of higher order. The lower inequality is essentially contained in [8], and a proof is sketched in [9]. For completeness we include detailed proofs of both inequalities here.

Proof of comparison lemma. After rotation, the problem reduces to proving the inequalities for points z in the real interval $0 \le z < 1$. (Let $U(r) = u(re^{i\theta})$ for fixed θ.) To prove the upper inequality $|u(x)| \le w(x)$ for $0 \le x < 1$, we convert the differential equation and initial conditions to an integral equation. Integration gives

$$u'(x) = 1 - \int_0^x p(t)u(t)\,dt \qquad \text{and}$$
$$u(x) = x - \int_0^x \int_0^y p(t)u(t)\,dt\,dy\,.$$

Reversing the order of integration, we have

$$u(x) = x - \int_0^x (x-t)p(t)u(t)\,dt\,, \qquad \text{so that}$$
$$|u(x)| \le x + \int_0^x (x-t)Q(t)|u(t)|\,dt\,, \qquad 0 \le x < 1\,.$$

A similar analysis gives

$$w(x) = x + \int_0^x (x-t)Q(t)w(t)\,dt\,, \qquad 0 \le x < 1\,.$$

Subtraction now shows that $h(x) = |u(x)| - w(x)$ satisfies

$$h(x) \le \int_0^x (x-t)Q(t)h(t)\,dt\,, \qquad 0 \le x < 1\,.$$

To infer that $h(x) \le 0$, fix an arbitrary point $x_0 \in (0,1)$ and let

$$s_0 = \sup\{s \in [0,1) : h(x) \le 0 \text{ for all } x \in [0,s]\}\,.$$

If $s_0 \ge x_0$, then $h(x) \le 0$ in $(0, x_0)$ and the proof is finished, since x_0 was chosen arbitrarily in $(0,1)$. If $s_0 < x_0$, let M be the maximum value of $Q(x)$ for $0 \le x \le x_0$ and choose $x_1 \in (s_0, x_0)$ such that $M(x_1 - s_0) < 1$. Let μ be the maximum value of $h(x)$ for $s_0 \le x \le x_1$, so that $\mu = h(x_2) > 0$ for some $x_2 \in (s_0, x_1]$. Then

$$\begin{aligned}\mu = h(x_2) &\le \int_0^{x_2} (x_2 - t)Q(t)h(t)\,dt \le \int_{s_0}^{x_2} (x_2 - t)Q(t)h(t)\,dt \\ &\le \int_{s_0}^{x_2} (x_2 - t)Q(t)\mu\,dt \le M(x_1 - s_0)\mu < \mu\,,\end{aligned}$$

a contradiction. This shows that $s_0 \ge x_0$, which proves that $h(x) \le 0$, or $w(x) \le |u(x)|$ in $[0, x_0)$, hence in $[0,1)$. Thus $w(|z|) \le |u(z)|$ for all $z \in \mathbb{D}$.

Now consider the lower bound $v(|z|) \le |u(z)|$ for $|z| < \xi$. Again it suffices to carry out the proof for $z \in [0,1)$. Let $\varphi(x) = |u(x)|$, so that $\varphi^2 = u\overline{u}$, and calculate

$$\varphi(x)\varphi'(x) = \tfrac{1}{2}\left[u'(x)\overline{u(x)} + u(x)\overline{u'(x)}\right] = \mathrm{Re}\left\{u'(x)\overline{u(x)}\right\}\,.$$

Hence $|\varphi'(x)| \le |u'(x)|$ wherever $u(x) \ne 0$. Another differentiation gives

$$\varphi(x)\varphi''(x) + \varphi'(x)^2 = \mathrm{Re}\left\{u''(x)\overline{u(x)}\right\} + |u'(x)|^2\,,$$

from which we infer that

$$\varphi(x)\varphi''(x) \ge \mathrm{Re}\left\{u''(x)\overline{u(x)}\right\} = -\mathrm{Re}\{p(x)\}\varphi(x)^2\,,$$

in view of the differential equation for u. Consequently, since $\varphi(x) = |u(x)| \ge 0$ and $|p(x)| \le Q(x)$, we arrive at the differential inequality

$$\varphi''(x) + Q(x)\varphi(x) \ge 0\,, \qquad 0 \le x < 1\,.$$

On the other hand, the function v satisfies the differential equation

$$v''(x) + Q(x)v(x) = 0\,, \qquad 0 \le x < 1\,.$$

Since $v(0) = \varphi(0) = 0$ and $v'(0) = \varphi'(0) > 0$, it now follows from the Sturm comparison theorem that $\varphi(x) \ge v(x)$ up to the first zero of v. Thus $|u(x)| \ge v(x)$ for $0 \le x < \xi$, and so $|u(z)| \ge v(|z|)$ for $|z| < \xi$. $\square$

§3. Distortion of analytic functions.

We turn now to the main result of this paper. It will be convenient to employ the notation $\Delta_f(\alpha,\beta)$ defined by (3), where f is analytic and locally univalent in the disk and $\alpha,\beta\in\mathbb{D}$. It is important to observe that this quantity is invariant under both precomposition and postcomposition with Möbius transformations. Specifically, if σ is any Möbius automorphism of the disk, then

$$\Delta_{f\circ\sigma}(\alpha,\beta)=\Delta_f(\sigma(\alpha),\sigma(\beta))\,,\qquad \alpha,\beta\in\mathbb{D}\,,$$

as can be seen by direct calculation using the identity

$$\frac{|\sigma'(z)|}{1-|\sigma(z)|^2}=\frac{1}{1-|z|^2}\,,\qquad z\in\mathbb{D}\,. \tag{6}$$

To show that

$$\Delta_{T\circ f}(\alpha,\beta)=\Delta_f(\alpha,\beta)$$

for every Möbius transformation T, it suffices to verify by simple calculation that $\Delta_{1/f}(\alpha,\beta)=\Delta_f(\alpha,\beta)$, since the relation clearly holds for every affine mapping T. Now for the main theorem.

Theorem 1. *Let f be analytic and locally univalent in $\mathbb{D}$ and suppose that $\|\mathcal{S}f\|\le 2(1+\delta^2)$ for some $\delta>0$. Then*

$$\Delta_f(\alpha,\beta)\ge\frac{1}{\delta}\,\sin(\delta\,d(\alpha,\beta)) \tag{7}$$

for all $\alpha,\beta\in\mathbb{D}$ with hyperbolic separation $d(\alpha,\beta)\le\pi/\delta$, and

$$\Delta_f(\alpha,\beta)\le\frac{1}{\sqrt{2+\delta^2}}\,\sinh\bigl(\sqrt{2+\delta^2}\,d(\alpha,\beta)\bigr) \tag{8}$$

for all $\alpha,\beta\in\mathbb{D}$. Each of the inequalities (7) and (8) is sharp; for each pair of points α and β in the specified range, equality occurs for some function f with $\|\mathcal{S}f\|\le 2(1+\delta^2)$. Equality holds in (7) precisely for $f=T\circ F\circ\sigma$ and in (8) for $f=T\circ G\circ\sigma$, where F and G are defined by

$$F(z)=\left(\frac{1+z}{1-z}\right)^{i\delta}\qquad\text{and}\qquad G(z)=\left(\frac{1+z}{1-z}\right)^{\sqrt{2+\delta^2}}\,, \tag{9}$$

σ is the Möbius automorphism of $\mathbb{D}$ with $\sigma(\alpha)=0$ and $\sigma(\beta)>0$, and T is an arbitrary Möbius transformation. For each such function f, equality holds along the entire (admissible portion of the) hyperbolic geodesic through α and β. Conversely, if either inequality holds for all points α and β in the specified range, then $\|\mathcal{S}f\|\le 2(1+\delta^2)$.

Proof. The strategy is to establish the inequalities first in the special case where $\alpha = 0$, then to derive them in the general case by Möbius invariance. Suppose that

$$|\mathcal{S}f(z)| \leq \frac{2(1+\delta^2)}{(1-|z|^2)^2}\,, \qquad z \in \mathbb{D}\,,$$

for some $\delta > 0$, and assume without loss of generality that $f(0) = 0$ and $f'(0) = 1$. Define

$$g(z) = -\frac{1}{f(z)}\,, \qquad \text{so that} \quad g'(z) = \frac{f'(z)}{f(z)^2}\,.$$

Then the function

$$u(z) = [g'(z)]^{-1/2} = z + c_2 z^2 + \dots$$

is analytic in $\mathbb{D}$, with $u(0) = 0$ and $u'(0) = 1$, and it satisfies the differential equation

$$u'' + [\tfrac{1}{2}\,\mathcal{S}f]\,u = 0\,,$$

since $\mathcal{S}g = \mathcal{S}f$. Define the functions $v(x)$ and $w(x)$ by

$$v''(x) + \frac{1+\delta^2}{(1-x^2)^2}\,v(x) = 0\,, \qquad v(0) = 0\,,\ v'(0) = 1 \qquad \text{and}$$

$$w''(x) - \frac{1+\delta^2}{(1-x^2)^2}\,w(x) = 0\,, \qquad w(0) = 0\,,\ w'(0) = 1\,.$$

Suppose that $v(x) > 0$ in the interval $(0, \xi)$, where $0 < \xi \leq 1$. Then in view of the hypothesis that $\left|\frac{1}{2}\mathcal{S}f(z)\right| \leq (1+\delta^2)(1-|z|^2)^{-2}$ in $\mathbb{D}$, we infer from the comparison lemma that $|u(z)| \leq w(|z|)$ for all $z \in \mathbb{D}$, and $v(|z|) \leq |u(z)|$ for all $z \in \mathbb{D}$ with $|z| < \xi$.

The solutions $v(x)$ and $w(x)$ are

$$v(x) = \frac{1}{\delta}\sqrt{1-x^2}\,\sin\left(\frac{\delta}{2}\log\frac{1+x}{1-x}\right)\,, \tag{10}$$

$$w(x) = \frac{\sqrt{1-x^2}}{\sqrt{2+\delta^2}}\,\sinh\left(\frac{\sqrt{2+\delta^2}}{2}\log\frac{1+x}{1-x}\right)\,. \tag{11}$$

These explicit formulas can be found with reference to Kamke [14], or by means of the substitution

$$y(t) = \frac{v(x)}{\sqrt{1-x^2}}\,, \qquad \text{where} \quad t = \frac{1}{2}\log\frac{1+x}{1-x}\,,$$

which reduces the first differential equation to $y''(t) + \delta^2 y(t) = 0$. Similarly, the second equation reduces to $y''(t) - (2+\delta^2)y(t) = 0$ through the same substitution with w in place of v.

The first positive zero of $v(x)$ occurs at the point $\xi = \tanh(\pi/\delta)$. Since

$$u(z) = [g'(z)]^{-1/2} = f(z)[f'(z)]^{-1/2},$$

the inequality $|u(z)| \geq v(|z|)$ obtained from the comparison lemma reduces to

$$\frac{|f(z)|^2}{|f'(z)|} \geq \frac{1}{\delta^2}(1-|z|^2)\sin^2\left(\frac{\delta}{2}\log\frac{1+|z|}{1-|z|}\right), \tag{12}$$

$$\text{or} \qquad \Delta_f(0,z) \geq \frac{1}{\delta}\sin\left(\delta\, d(0,z)\right) \qquad \text{for} \;\; d(0,z) \leq \frac{\pi}{\delta}.$$

Now let α and β be arbitrary points in the unit disk and define

$$f_1(z) = \frac{f(\sigma(z)) - f(\alpha)}{(1-|\alpha|^2)f'(\alpha)}, \qquad \text{where} \;\; \sigma(z) = \frac{z+\alpha}{1+\overline{\alpha}z}. \tag{13}$$

This function has the form $f_1 = T \circ f \circ \sigma$, where T is a Möbius transformation, and so

$$\Delta_{f_1}(0,z) = \Delta_{f\circ\sigma}(0,z) = \Delta_f(\sigma(0),\sigma(z)) = \Delta_f(\alpha,\sigma(z)).$$

On the other hand, $\mathcal{S}f_1 = \mathcal{S}(f\circ\sigma) = [(\mathcal{S}f)\circ\sigma]\sigma'^2$, so that

$$|\mathcal{S}f_1(z)| = |\mathcal{S}f(\sigma(z))||\sigma'(z)|^2 \leq \frac{2(1+\delta^2)|\sigma'(z)|^2}{(1-|\sigma(z)|^2)^2} = \frac{2(1+\delta^2)}{(1-|z|^2)^2}.$$

Since $\|\mathcal{S}f_1\| \leq 2(1+\delta^2)$ and $f_1(0) = 0$, $f_1'(0) = 1$, it follows from what has already been proved that

$$\Delta_{f_1}(0,z) \geq \frac{1}{\delta}\sin\left(\delta\, d(0,z)\right), \qquad d(0,z) \leq \frac{\pi}{\delta}.$$

Therefore, if z is chosen so that $\sigma(z) = \beta$, we have

$$\Delta_f(\alpha,\beta) = \Delta_{f_1}(0,z) \geq \frac{1}{\delta}\sin\left(\delta\, d(\sigma(0),\sigma(z)))\ \right) = \frac{1}{\delta}\sin\left(\delta\, d(\alpha,\beta)\right)$$

for $d(\alpha,\beta) \leq \pi/\delta$, by the Möbius invariance of the hyperbolic metric. The proof of the lower bound (7) is now complete.

The upper bound is derived in similar fashion. The comparison lemma gives $|u(z)| \leq w(|z|)$ for all $z \in \mathbb{D}$, which reduces to

$$\Delta_f(0,z) \leq \frac{1}{\sqrt{2+\delta^2}}\sinh\left(\sqrt{2+\delta^2}\, d(0,z)\right).$$

It then follows as before that

$$\Delta_f(\alpha,\beta) \leq \frac{1}{\sqrt{2+\delta^2}}\sinh\left(\sqrt{2+\delta^2}\, d(\alpha,\beta)\right), \qquad \alpha,\beta \in \mathbb{D},$$

by choosing $z = \sigma^{-1}(\beta)$. This proves (8).

In order to prove the sharpness of (7), we now show that for each pair of points $\alpha, \beta \in \mathbb{D}$ with $0 < d(\alpha,\beta) < \pi/\delta$, there is a function f with $\|\mathcal{S}f\| \le 2(1+\delta^2)$ such that $\Delta_f(\alpha,\beta) = \frac{1}{\delta}\sin(\delta\, d(\alpha,\beta))$. By Möbius invariance, it is equivalent to show that $\Delta_F(0,b) = \frac{1}{\delta}\sin(\delta\, d(0,b))$, where $F = f\circ\sigma^{-1}$ and σ is the Möbius automorphism of the disk for which $\sigma(\alpha) = 0$ and $\sigma(\beta) = b > 0$. This will be the case if and only if $\mathcal{S}F(z) = 2(1+\delta^2)(1-z^2)^{-2}$, which is the requirement for equality in the comparison lemma (*cf.* [8]). Thus the general form of the extremal function is $f = T\circ F\circ\sigma$, where F is a particular function (as given by (9), for instance) with Schwarzian $\mathcal{S}F(z) = 2(1+\delta^2)(1-z^2)^{-2}$, σ is the Möbius automorphism defined above, and T is an arbitrary Möbius transformation. Similarly, for each pair of distinct points $\alpha,\beta\in\mathbb{D}$, equality occurs in (8) precisely for functions of the form $f = T\circ G\circ\sigma$, where G is a particular function (as defined by (9), for instance) with $\mathcal{S}G(z) = -2(1+\delta^2)(1-z^2)^{-2}$, σ is the Möbius automorphism with $\sigma(\alpha) = 0$ and $\sigma(\beta) > 0$, and T is an arbitrary Möbius transformation (*cf.* [12]).

Conversely, we want to show that either of the two-point distortion conditions (7) or (8) implies the bound $\|\mathcal{S}f\| \le 2(1+\delta^2)$ on the Schwarzian norm. The proofs follow an argument given by Chuaqui and Pommerenke [9] to show that the condition (3) implies $\|\mathcal{S}f\| \le 2$. It will suffice to carry out the details only for the condition (8), because the proof for (7) is quite similar. In view of the Möbius invariance, no information is lost if we take $\alpha = 0$. Without loss of generality, we may assume that $f(0) = 0$ and $f'(0) = 1$, so that

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots .$$

The condition (8) then reduces to

$$\frac{|f(z)|^2}{|f'(z)|} \le \frac{1-|z|^2}{2+\delta^2}\,\sinh^2\!\left(\sqrt{2+\delta^2}\, d(0,z)\right), \qquad z\in\mathbb{D}. \tag{14}$$

In order to conclude from (14) that $\|\mathcal{S}f\| \le 2(1+\delta^2)$, it will suffice to show that $|\mathcal{S}f(0)| \le 2(1+\delta^2)$, because of the Möbius invariance. Indeed, for the function f_1 defined by (13) we have

$$(1-|z|^2)^2|\mathcal{S}f_1(z)| = (1-|\sigma(z)|^2)^2|\mathcal{S}f(\sigma(z))|,$$

and so $|\mathcal{S}f_1(0)| = (1-|\alpha|^2)^2|\mathcal{S}f(\alpha)|$. But $\mathcal{S}f(0) = 6(a_3 - a_2^2)$, so the problem reduces to showing that $|a_3 - a_2^2| \le \frac{1}{3}(1+\delta^2)$. Straightforward calculations give

$$\frac{f(z)^2}{f'(z)} = z^2[1 + (a_2^2 - a_3)z^2 + \dots] \qquad \text{and}$$

$$\frac{1-|z|^2}{2+\delta^2}\,\sinh^2\!\left(\sqrt{2+\delta^2}\, d(0,z)\right) = r^2[1 + \tfrac{1}{3}(1+\delta^2)r^2 + \dots], \qquad r = |z|.$$

Therefore, the inequality (14) implies

$$\begin{aligned} &\bigl|1+(a_2^2-a_3)z^2+O(r^3)\bigr|^2 \le \bigl|1+\tfrac{1}{3}(1+\delta^2)r^2+O(r^3)\bigr|^2\,, \\ \text{or}\qquad &1+2\mathrm{Re}\,\bigl\{(a_2^2-a_3)z^2+O(r^3)\bigr\} \le 1+\tfrac{2}{3}(1+\delta^2)r^2+O(r^3)\,, \end{aligned}$$

from which we infer that

$$\mathrm{Re}\,\bigl\{(a_2^2-a_3)e^{2i\theta}\bigr\} \le \tfrac{1}{3}(1+\delta^2)$$

by setting $z=re^{i\theta}$ for fixed θ and letting $r\to 0$. Since the angle θ can be chosen arbitrarily, we conclude that $|a_3-a_2^2|\le \frac{1}{3}(1+\delta^2)$, as desired.

Essentially the same calculations show that if the inequality (12) holds for all $z\in\mathbb{D}$ with $d(0,z)\le \pi/\delta$ (or equivalently for $|z|\le \tanh(\pi/\delta)$), then $|\mathcal{S}f(0)|\le 2(1+\delta^2)$ and so $\|\mathcal{S}f\|\le 2(1+\delta^2)$. $\square$

Similar results are obtained under the hypothesis $\|\mathcal{S}f\|\le 2(1-\delta^2)$ for $0<\delta<1$. Then the relevant functions v and w of the comparison lemma are obtained by replacing δ by $i\delta$ in the formulas (10) and (11). Specifically,

$$\begin{aligned} v(x) &= \frac{1}{\delta}\sqrt{1-x^2}\,\sinh\left(\frac{\delta}{2}\log\frac{1+x}{1-x}\right)\,, \\ w(x) &= \frac{\sqrt{1-x^2}}{\sqrt{2-\delta^2}}\,\sinh\left(\frac{\sqrt{2-\delta^2}}{2}\log\frac{1+x}{1-x}\right)\,. \end{aligned}$$

The inequalities $v(|z|)\le |u(z)|\le w(|z|)$ now reduce to

$$\frac{1}{\delta}\,\sinh(\delta\, d(0,z)) \le \Delta_f(0,z) \le \frac{1}{\sqrt{2-\delta^2}}\,\sinh\bigl(\sqrt{2-\delta^2}\,d(0,z)\bigr)\,, \qquad z\in\mathbb{D}\,,$$

whereupon the same argument based on Möbius invariance gives

$$\frac{1}{\delta}\,\sinh(\delta\, d(\alpha,\beta)) \le \Delta_f(\alpha,\beta) \le \frac{1}{\sqrt{2-\delta^2}}\,\sinh\bigl(\sqrt{2-\delta^2}\,d(\alpha,\beta)\bigr) \tag{15}$$

for all $\alpha,\beta\in\mathbb{D}$. Conversely, if either of the inequalities in (15) holds for some $\delta\in(0,1)$ and for all α and β in $\mathbb{D}$, calculations similar to the above lead to the conclusion that $\|\mathcal{S}f\|\le 2(1-\delta^2)$.

Theorem 1 was essentially proved by D. Mejía [15] and was discovered independently in joint work by M. Chuaqui, P. Duren, and B. Osgood.

§4. Distortion of harmonic mappings.

By a similar method, the lower bound (7) can be extended to harmonic mappings, or rather to their canonical lifts to minimal surfaces. The result will generalize a theorem in the paper [6] for the special case of the extremal Nehari function $p(x) = (1-x^2)^{-2}$. As in [6], we begin with a distortion theorem for curves in $\mathbb{R}^n$.

Let $\varphi : (-1,1) \mapsto \mathbb{R}^n$ be a mapping of class C^3 with $\varphi'(x) \neq 0$. The Ahlfors Schwarzian of φ is defined by

$$S_1\varphi = \frac{\langle \varphi', \varphi''' \rangle}{|\varphi'|^2} - 3\frac{\langle \varphi', \varphi'' \rangle^2}{|\varphi'|^4} + \frac{3}{2}\frac{|\varphi''|^2}{|\varphi'|^2}\,,$$

where $\langle \cdot\,, \cdot \rangle$ denotes the Euclidean inner product and $|\mathbf{x}|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$ for $\mathbf{x} \in \mathbb{R}^n$. As Ahlfors [1] observed, it is invariant under postcomposition with Möbius transformations of $\mathbb{R}^n$. Chuaqui and Gevirtz [7] used it to give an injectivity criterion for curves. Here is a special case of their theorem.

Theorem A. *Let $\varphi : (-1,1) \mapsto \mathbb{R}^n$ be a curve of class C^3 with tangent vector $\varphi'(x) \neq 0$. If $S_1\varphi(x) \le 2(1-x^2)^{-2}$, then φ is injective.*

Chuaqui and Gevirtz also showed that the arclength $s = s(x)$ of the curve φ has Schwarzian

$$\mathcal{S}s(x) = S_1\varphi(x) - \frac{1}{2}|\varphi'(x)|^2\kappa(x)^2 \le S_1\varphi(x)\,, \tag{16}$$

where $\kappa = \kappa(x)$ is the curvature of φ.

Our next theorem extends Theorem A to a criterion for uniform local injectivity, in the manner of B. Schwarz' extension of Nehari's theorem. Moreover, it expresses the local injectivity in quantitative form as a two-point distortion result analogous to the lower bound (7) in Theorem 1. In terms of the curve $\varphi(x)$, we define

$$\Delta_\varphi(a,b) = \frac{|\varphi(a) - \varphi(b)|}{\{(1-a^2)|\varphi'(a)|\}^{1/2}\,\{(1-b^2)|\varphi'(b)|\}^{1/2}}\,, \qquad a,b \in (-1,1)\,.$$

We are now prepared to state the theorem.

Theorem 2. *Let $\varphi : (-1,1) \mapsto \mathbb{R}^n$ be a curve of class C^3 with $\varphi'(x) \neq 0$. If*

$$S_1\varphi(x) \le \frac{2(1+\delta^2)}{(1-x^2)^2} \qquad \text{for some} \quad \delta > 0\,,$$

then the inequality

$$\Delta_\varphi(a,b) \ge \frac{1}{\delta}\,\sin(\delta\, d(a,b)) \tag{17}$$

holds for all $a,b \in (-1,1)$ with $d(a,b) \le \pi/\delta$.

Proof. First observe that the quantity $\Delta_\varphi(a,b)$ is again Möbius invariant. If σ is any Möbius automorphism of the disk that preserves the real segment $(-1,1)$, or equivalently if σ is a Möbius automorphism with real coefficients, then

$$\Delta_{\varphi\circ\sigma}(a,b) = \Delta_\varphi(\sigma(a),\sigma(b))\,, \qquad a,b \in (-1,1)\,.$$

If T is any Möbius transformation of $\mathbb{R}^n$, then $\Delta_{T\circ\varphi}(a,b) = \Delta_\varphi(a,b)$. The proofs for curves are essentially the same as for analytic functions.

As in the proof of Theorem 1, we will derive the inequality (17) first for $a = 0$, then deduce the general result by Möbius invariance. Because of Möbius invariance, we may assume without loss of generality that $\varphi(0) = 0$ and $|\varphi'(0)| = 1$. Consider the inverted curve

$$\Phi(x) = \frac{\varphi(x)}{|\varphi(x)|^2}\,, \qquad \text{with } \; |\Phi'(x)| = \frac{|\varphi'(x)|}{|\varphi(x)|^2}\,,$$

as a straightforward calculation of $|\Phi'(x)|^2$ shows. By Möbius invariance, $S_1\Phi = S_1\varphi$. Recall that if $g(x)$ is a real-valued function with $g'(x) > 0$, the function $u(x) = g'(x)^{-1/2}$ satisfies the differential equation $u'' + \frac{1}{2}(\mathcal{S}g)u = 0$. Thus if we take $g(x) = s(x)$, the arclength function along the curve $\Phi(x)$, we see that the function

$$u(x) = |\Phi'(x)|^{-1/2} = \frac{|\varphi(x)|}{|\varphi'(x)|^{1/2}}$$

satisfies $u'' + \frac{1}{2}(\mathcal{S}s)u = 0$ and has initial data $u(0) = 0$ and $u'(0) = 1$, since $\varphi(0) = 0$ and $|\varphi'(0)| = 1$. But

$$\mathcal{S}s(x) \le S_1\Phi(x) = S_1\varphi(x) \le \frac{2(1+\delta^2)}{(1-x^2)^2}\,,$$

so it follows from the Sturm comparison theorem that $u(x) \ge v(x)$ for $0 \le x \le \tanh(\pi/\delta)$, where $v(x)$ is the function given in (10). In terms of the hyperbolic metric, this last inequality takes the form

$$\Delta_\varphi(0,x) \ge \frac{1}{\delta}\,\sin(\delta\, d(0,x))\,, \qquad d(0,x) \le \pi/\delta\,,$$

which is the desired result (17) for $a = 0$. The general inequality (17) is deduced from this special case by Möbius invariance. □

With the help of Theorem 2, we can now derive a two-point distortion inequality for the canonical lift of a harmonic mapping to a minimal surface. A harmonic mapping is a complex-valued harmonic function $f(z) = u(z) + iv(z)$, for $z = x + iy$ in the unit disk $\mathbb{D}$ of the complex plane. Such a mapping has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ and $g(0) = 0$. The basic properties of harmonic mappings are described in [11].

According to the Weierstrass–Enneper formulas, a harmonic mapping $f = h + \overline{g}$ with $|h'(z)| + |g'(z)| \ne 0$ lifts locally to a minimal surface described by conformal parameters if and only if its dilatation $\omega = g'/h'$ has the form $\omega = q^2$ for some meromorphic function q. The Cartesian coordinates (U, V, W) of the surface are then given by

$$U(z) = \mathrm{Re}\{f(z)\}\,, \quad V(z) = \mathrm{Im}\{f(z)\}\,, \quad W(z) = 2\,\mathrm{Im}\left\{\int_0^z h'(\zeta)q(\zeta)\,d\zeta\right\}.$$

We use the notation $\widetilde{f}(z) = \big(U(z), V(z), W(z)\big)$ for the lifted mapping from $\mathbb{D}$ to the minimal surface. The first fundamental form of the surface is $ds^2 = \lambda^2|dz|^2$, where the conformal metric is $\lambda = |h'| + |g'|$.

For a harmonic mapping $f = h + \overline{g}$ with $\lambda(z) = |h'(z)| + |g'(z) > 0$, whose dilatation is the square of a meromorphic function, the Schwarzian derivative is defined by the formula

$$\mathcal{S}f = 2\big(\sigma_{zz} - {\sigma_z}^2\big)\,, \qquad \sigma = \log\lambda\,.$$

If f is analytic, it is easily verified that $\mathcal{S}f$ reduces to the classical Schwarzian.

In a previous paper [3], the following criterion was given for the lift of a harmonic mapping to be univalent.

Theorem B. *Let $f = h + \overline{g}$ be a harmonic mapping of the unit disk, with $\lambda(z) = |h'(z)| + |g'(z)| > 0$ and dilatation $g'/h' = q^2$ for some meromorphic function q. Let $\widetilde{f}$ denote the Weierstrass–Enneper lift of f to a minimal surface with Gauss curvature $K = K(\widetilde{f}(z))$ at the point $\widetilde{f}(z)$. Suppose that the inequality*

$$|\mathcal{S}f(z)| + \lambda(z)^2|K(\widetilde{f}(z))| \le \frac{2}{(1-|z|^2)^2}$$

holds for all $z \in \mathbb{D}$. Then $\widetilde{f}$ is univalent in $\mathbb{D}$.

If f is analytic, its associated minimal surface is the complex plane itself, with Gauss curvature $K = 0$, and the result reduces to Nehari's theorem.

In the paper [6], Theorem B was sharpened to express the univalence in the form of a two-point distortion condition. It was shown in [4] that if the bound $2(1-|z|^2)^{-2}$ is weakened to $2(1+\delta^2)(1-|z|^2)^{-2}$, then $\widetilde{f}$ is uniformly locally univalent, the analogue of B. Schwarz' extension of Nehari's theorem. We now express the uniform local univalence in quantitative form, thus obtaining a harmonic analogue of the lower bound (7) in Theorem 1. Let

$$\Delta_{\widetilde{f}}(\alpha,\beta) = \frac{|\widetilde{f}(\alpha) - \widetilde{f}(\beta)|}{\{(1-|\alpha|^2)\lambda(\alpha)\}^{1/2}\,\{(1-|\beta|^2)\lambda(\beta)\}^{1/2}}\,, \qquad \alpha,\beta \in \mathbb{D}\,,$$

where λ is the conformal metric of the minimal surface. With this notation, we are prepared to state the theorem.

Theorem 3. *Let $f = h + \overline{g}$ be a harmonic mapping of the unit disk, with $\lambda(z) = |h'(z)| + |g'(z)| > 0$ and dilatation $g'/h' = q^2$ for some meromorphic function q. Let $\widetilde{f}$ denote the canonical lift of f to a minimal surface. Suppose that*

$$|\mathcal{S}f(z)| + \lambda(z)^2|K(\widetilde{f}(z))| \le \frac{2(1+\delta^2)}{(1-|z|^2)^2}\,, \qquad z \in \mathbb{D}\,. \tag{18}$$

Then

$$\Delta_{\widetilde{f}}(\alpha,\beta) \ge \frac{1}{\delta}\,\sin(\delta\, d(\alpha,\beta)) \tag{19}$$

for all $\alpha, \beta \in \mathbb{D}$ with hyperbolic separation $d(\alpha,\beta) \le \pi/\delta$. For each pair of points α, β with $0 < d(\alpha,\beta) < \pi/\delta$, equality occurs in (19) *only for harmonic mappings of the form $f = h + c\,\overline{h}$, with c a constant of modulus $|c| < 1$ and $h = T \circ F \circ \sigma$, where F is defined by* (9)*, σ is the Möbius automorphism of $\mathbb{D}$ for which $\sigma(\alpha) = 0$ and $\sigma(\beta) > 0$, and T is an arbitrary Möbius transformation. The corresponding minimal surface is then a plane.*

Proof. The proof will apply Theorem 2. The canonical lift $\widetilde{f}$ onto a minimal surface Σ defines a curve $\widetilde{f} : (-1,1) \to \Sigma \subset \mathbb{R}^3$. As shown in [3], the Ahlfors Schwarzian of this curve satisfies

$$\begin{aligned} S_1\widetilde{f}(x) &= \mathrm{Re}\{\mathcal{S}f(x)\} + \tfrac{1}{2}\lambda(x)^2\kappa_e(\widetilde{f}(x))^2 + \tfrac{1}{2}\lambda(x)^2|K(\widetilde{f}(x))| \\ &\le \mathrm{Re}\{\mathcal{S}f(x)\} + \lambda(x)^2|K(\widetilde{f}(x))| \\ &\le |\mathcal{S}f(x)| + \lambda(x)^2|K(\widetilde{f}(x))|\,, \qquad -1 < x < 1\,, \end{aligned} \tag{20}$$

where $\kappa_e(\widetilde{f}(x))$ denotes the normal curvature of the curve at the point $\widetilde{f}(x)$. Thus the hypothesis (18) tells us that $S_1\widetilde{f}(x) \le 2(1+\delta^2)(1-x^2)^{-2}$, and so by Theorem 2 we have the inequality

$$\Delta_{\widetilde{f}}(a,b) \ge \frac{1}{\delta}\,\sin(\delta\, d(a,b)) \tag{21}$$

for all $a, b \in (-1,1)$ with $d(a,b) \le \pi/\delta$, since $|\widetilde{f}'(x)| = \lambda(x)$.

In order to extend the inequality (21) to arbitrary points $\alpha, \beta \in \mathbb{D}$, we appeal again to Möbius invariance. Observe first that the quantity $\Delta_{\widetilde{f}}(\alpha,\beta)$ is invariant under precomposition with Möbius automorphisms of the disk. Indeed, if σ is any such automorphism, the composition $F = f \circ \sigma$ is a harmonic mapping with canonical lift $\widetilde{F} = \widetilde{f} \circ \sigma$ and conformal metric $\Lambda(z) = \lambda(\sigma(z))|\sigma'(z)|$. Combining this with the identity (6), we see that $\Delta_{\widetilde{F}}(\alpha,\beta) = \Delta_{\widetilde{f}}(\sigma(\alpha),\sigma(\beta))$. Given any pair of points $\alpha, \beta \in \mathbb{D}$, choose σ so that $\sigma(a) = \alpha$ and $\sigma(b) = \beta$ for some $a, b \in (-1,1)$. In view of (6), the hypothesis (18) is also Möbius invariant, and so $\Delta_{\widetilde{F}}(a,b) \ge \frac{1}{\delta}\,\sin(\delta\, d(a,b))$, by what we have already proved. But $d(a,b) = d(\alpha,\beta)$ by Möbius invariance of the hyperbolic metric, whereas

$$\Delta_{\widetilde{F}}(a,b) = \Delta_{\widetilde{f}}(\sigma(a),\sigma(b)) = \Delta_{\widetilde{f}}(\alpha,\beta)\,.$$

Therefore, the inequality (19) holds for all points $\alpha, \beta \in \mathbb{D}$ with $d(\alpha,\beta) \le \pi/\delta$.

We now turn to the case of equality in (19) for two distinct points $\alpha, \beta \in \mathbb{D}$ with $d(\alpha,\beta) < \pi/\delta$. After precomposing with an automorphism of the disk, we may assume that $\alpha = 0$ and $\beta = r$ with $0 < r < \pi/\delta$. More precisely, if σ is the automorphism with $\sigma(\alpha) = 0$ and $\sigma(\beta) = r > 0$, we need only consider equality for functions $f_1 = f \circ \sigma^{-1}$ at the points 0 and r. Let $\varphi(x) = \widetilde{f}_1(x)$ denote the lifted curve on the corresponding minimal surface Σ. With the notation in the proof of Theorem 2, we see that equality in (19), namely $\Delta_{\widetilde{f}_1}(0,r) = \frac{1}{\delta}\,\sin(\delta\, d(0,r))$, is equivalent to $u(r) = \frac{1}{\delta}\,\sin(\delta\, d(0,r))$, which by the Sturm comparison theorem can occur only if

$$\mathcal{S}s(x) = S_1\varphi(x) = \frac{2(1+\delta^2)}{(1-x^2)^2} \qquad \text{for all} \ \ x \in [0,r]\,. \tag{22}$$

But in view of (16), the equality $\mathcal{S}s(x) = S_1\varphi(x)$ implies that the curvature $\kappa(x)$ of the curve φ vanishes for all $x \in [0,r]$, and so that portion of the curve is a straight line in space. On the other hand, because of (20) and the hypothesis (18), the equality $S_1\varphi(x) = 2(1+\delta^2)(1-x^2)^{-2}$ implies that the normal curvature has the property $\kappa_e(\varphi(x))^2 \equiv |K(\varphi(x))|$ on $[0,r]$, so that the corresponding portion of the curve is a line of curvature of Σ. (Here we use the fact that Σ is a minimal surface, with zero mean curvature.) But by uniqueness in the Björling problem (*cf.* [10]), a minimal surface containing a straight line segment as a line of curvature must reduce to a plane. Therefore, as shown in [2], the harmonic mapping f_1 has the form $h_1 + c\,\overline{h_1}$ for some locally univalent analytic function h_1 and some constant c with $|c| < 1$. It is then easily seen that $\mathcal{S}f_1 = \mathcal{S}h_1$. Furthermore, since the surface Σ is a plane, it has Gauss curvature $K = 0$, and so (22) combines with (20) and (18) to show that

$$\mathcal{S}h_1(x) = \mathcal{S}f_1(x) = S_1\widetilde{f}_1(x) = \frac{2(1+\delta^2)}{(1-x^2)^2} \qquad \text{for all} \ \ x \in [0,r]\,.$$

But $\mathcal{S}h_1$ is an analytic function, so this implies that $\mathcal{S}h_1(z) = 2(1+\delta^2)(1-z^2)^{-2}$ for all $z \in \mathbb{D}$. Therefore, $h_1 = T \circ F$, where T is a Möbius transformation and F is a particular function (as given by (9), for instance) with Schwarzian $\mathcal{S}F(z) = 2(1+\delta^2)(1-z^2)^{-2}$. Hence $f = f_1 \circ \sigma = h + c\,\overline{h}$, where $h = T \circ F \circ \sigma$, as claimed. The argument also shows, as in Theorem 1, that the same functions f give equality along the entire hyperbolic geodesic through α and β. □

References

1. L. V. Ahlfors, *Cross-ratios and Schwarzian derivatives in* $\mathbb{R}^n$, Complex Analysis: Articles dedicated to Albert Pfluger on the occasion of his 80th birthday (J. Hersch and A. Huber, editors), Birkhäuser Verlag, Basel, 1988, pp. 1–15.
2. M. Chuaqui, P. Duren, and B. Osgood, *The Schwarzian derivative for harmonic mappings*, J. Analyse Math. **91** (2003), 329-351.
3. M. Chuaqui, P. Duren, and B. Osgood, *Univalence criteria for lifts of harmonic mappings to minimal surfaces*, J. Geom. Analysis **17** (2007), 49–74.
4. M. Chuaqui, P. Duren, and B. Osgood, *Schwarzian derivative criteria for valence of analytic and harmonic mappings*, Math. Proc. Cambridge Philos. Soc. **143** (2007), 473–486.
5. M. Chuaqui, P. Duren, and B. Osgood, *Schwarzian derivatives and uniform local univalence*, Comput. Methods Funct. Theory **8** (2008), 21–34.
6. M. Chuaqui, P. Duren, and B. Osgood, *Two-point distortion theorems for harmonic mappings*, Illinois J. Math., to appear.

7. M. Chuaqui and J. Gevirtz, *Simple curves in $\mathbb{R}^n$ and Ahlfors' Schwarzian derivative*, Proc. Amer. Math. Soc. **132** (2004), 223–230.
8. M. Chuaqui and B. Osgood, *Sharp distortion theorems associated with the Schwarzian derivative*, J. London Math. Soc. **48** (1993), 289–298.
9. M. Chuaqui and Ch. Pommerenke, *Characteristic properties of Nehari functions*, Pacific J. Math. **188** (1999), 83–94.
10. U. Dierkes, S. Hildebrandt, A. Küster, and O. Wohlrab, *Minimal Surfaces I. Boundary Value Problems*, Springer-Verlag, Berlin–Heidelberg, 1992.
11. P. Duren, *Harmonic Mappings in the Plane*, Cambridge University Press, Cambridge, U. K., 2004.
12. M. Essén and F. R. Keogh, *The Schwarzian derivative and estimates of functions analytic in the unit disc*, Math. Proc. Cambridge Philos. Soc. **78** (1975), 501–511.
13. H. Herold, *Ein Vergleichssatz für komplexe lineare Differentialgleichungen*, Math. Z. **126** (1972), 91–94.
14. E. Kamke, *Differentialgleichungen: Lösungsmethoden und Lösungen, Band 1: Gewöhnliche Differentialgleichungen*, 3. Auflage, Becker & Erler, Leipzig, 1944; reprinted by Chelsea Publishing Co., New York, 1948.
15. D. Mejía, *Invariancia bajo transformaciones de Möbius*, "Habilitation" thesis, Universidad Nacional de Colombia, Sede Medellín, 2007.
16. Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.
17. B. Schwarz, *Complex nonoscillation theorems and criteria of univalence*, Trans. Amer. Math. Soc. **80** (1955), 159–186.

Facultad de Matemáticas, P. Universidad Católica de Chile, Casilla 306, Santiago 22, Chile

E-mail address: mchuaqui@mat.puc.cl

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109–1043

E-mail address: duren@umich.edu

School of Integrated Studies, Pennsylvania College of Technology, Williamsport, PA 17701

E-mail address: wma@pct.edu

Departamento de Matemáticas, Universidad Nacional, A.A. 3840 Medellín, Colombia

E-mail address: dmejia@unal.edu.co

Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221-0025

E-mail address: minda@ucmail.uc.edu

Department of Electrical Engineering, Stanford University, Stanford, California 94305

E-mail address: osgood@ee.stanford.edu